**MATH1520C University Mathematics for Applications** **Spring 2021**

# Chapter 1: Notation and Functions

**Learning Objectives**:
(1) Identify the domain of a function, and evaluate a function from an equation.
(2) Gain familiarity with piecewise functions.
(3) Study the vertical line test.
(4) Know how to form and use composite functions.

## 1.1 Set

- **Set** is a collection of objects (called **elements**)
  1. Order of elements does not matter. E.g. $\{1,2,3\}=\{3,2,1\}$.
  2. Representation of a set is not unique. E.g. $\{-2,2\}=\{x\,|\,x^2=4\}$.
- $\in$: belongs to. If $a$ is an element of $A$, we say that $a$ belongs to $A$; denoted as $a\in A$.
- $\subset$: subset of. Let $A,B$ be two sets such that $\forall a\in A$, $a\in B$. Then we say that $A$ is a subset of $B$; denoted as $A\subset B$.

*Remark.* $A\subset B$ is sometimes written as $A\subseteq B$ to emphasize the fact that $A=B$ is a possibility. If $A\subset B$ but $A\neq B$, then $A$ is said to be a *proper subset* (or a strict subset) of $B$, written as $A\Subset B$.

$A\subset B\Leftrightarrow B\supset A$: $B$ is a *supset* of $A$.

**Example 1.1.1.**

1. $A=\{1,2,3\}$, $B=\{2,3,5,7\}$, $C=\{1,2,3,4,5\}$. Then $A\subseteq C$ (in fact $A\Subset C$), $1\in A$, but $1\notin B$ and $B\not\subseteq C$.
2. $C=$ the set of all students studying at CUHK. $M=$ the set of all math major students currently studying at CUHK. Then $M\subseteq C$. You $\in C$.

**Example 1.1.2.** Some important number sets:

1. $\mathbb{N}$: the set of all natural numbers (positive integers) $=\{1,2,3,\ldots\}$.
2. $\mathbb{Z}$: the set of all integers $=\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$.
3. $\mathbb{Q}$: the set of all rational numbers $=\{\frac{a}{b}\,|\,a,b\in\mathbb{Z},b\neq 0\}$.

4. $\mathbb{R}$: the set of all real numbers.

*Remark.* If the elements in a set can be ordered and the ordering are taken into account in the definition, then it is called an *ordered set*. E.g. $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ may call be viewed as ordered sets.

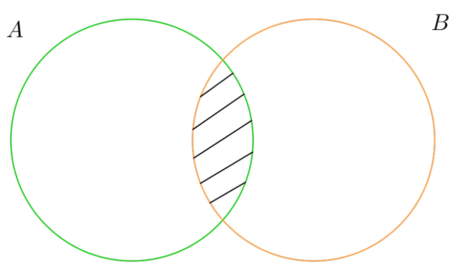
## 1.2 Intervals

- $[a, b] = \{x \mid a \leq x \leq b\}$. (closed interval)
- $(a, b) = \{x \mid a < x < b\}$. (open interval)
- $(a, b] = \{x \mid a < x \leq b\}$.
- $[a, \infty)$: the set of all real numbers $x$ such that $a \leq x$.

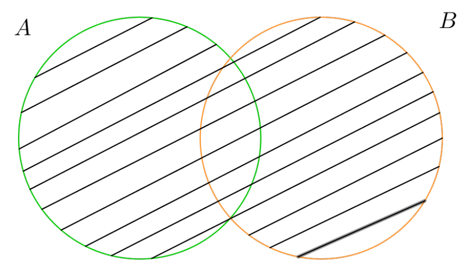
**Drawing open/closed intervals on the real line:**

## 1.3 Set operations

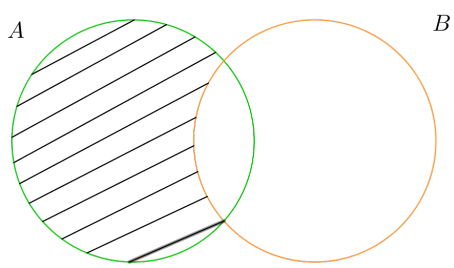
Let $A$, $B$ be two sets:



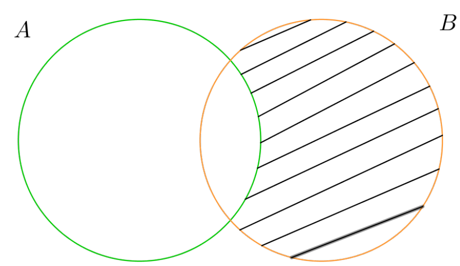

Intersection
$A \cap B = \{x \,|\, x \in A \;\; \text{and} \;\; x \in B\}$




Union
$A \cup B = \{x \,|\, x \in A \;\; \text{or} \;\; x \in B\}$




Relative complement of $B$ in $A$
$A \backslash B = \{x \,|\, x \in A \;\; \text{and} \;\; x \notin B\}$




Relative complement of $A$ in $B$
$B \backslash A = \{x \,|\, x \in B \;\; \text{and} \;\; x \notin A\}$

**Example 1.3.1.**

1. Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, $C = \{5\}$.
   $A \cap B = \{2, 3\}$, $A \cup B = \{1, 2, 3, 4\}$, $A \backslash B = \{1\}$, $B \backslash A = \{4\}$, $A \backslash C = A$.
2. $\mathbb{R} \backslash \{a\}$: the set of all real numbers $x$, except $x = a$.

*Exercise* 1.3.1.

1. What are the meanings of the following sets
   (a) $(-\infty, a)$.
   (b) $\mathbb{R} \backslash \{1, 2, 3\}$
   (c) $\mathbb{R} \backslash [2, 3)$.
2. Show that $\mathbb{R} \backslash [1, \infty) = (-\infty, 1)$.

## 1.4 Functions

**Definition 1.4.1.** A **function** is a rule that assigns to EACH element $x$ in a set $A$ EXACTLY ONE element $y$ in a set $B$. If the function is denoted by $f$, then we may write

$$f : A \to B.$$

The set $A$ is called the domain of the function. The set $B$ is called the codomain of $f$. The assigned elements in $B$ is called the range of $f$.

$x$ is the independent variable of $f$, and $y$ is the dependent variable of $f$.

Given $a \in A$, $f(a) \in B$ is said to be the *value* of the function $f$ at $a$. Given $S \subset A$,

$$f(S) := \{f(a) \,|\, a \in S\}$$

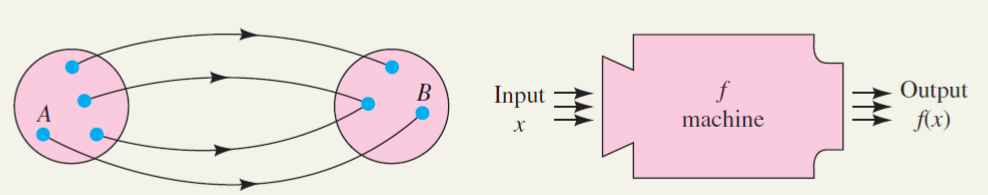
is said to be the *image* of $S$ under $f$. In particular, the "range" of $f$, as defined above, is $f(A) \subset B$.

When the domain and range of a function are both sets of real numbers, the function is said to be a real-valued function of one variable, and we write

$$f : \mathbb{R} \to \mathbb{R}.$$

Most functions encountered in this course are real-valued functions of one variable. Unless otherwise specified, a function is a real-valued function of one variable in this course.

*Remark.* There is some ambiguity in the definition of "range" in math literature. See the Wiki article. A function $f : A \to B$ is also called a *map from $A$ to $B$*; $A$ is the *source* of $f$ and $B$ is the *target* of $f$.




(a) A function as a mapping (b) A function as a machine

**Example 1.4.1.** $f : [-1, 3) \to \mathbb{R}$ is defined by $f(x) = x^2 + 4$ (sometimes written as $y = x^2 + 4$). Then

$$f(0) = (0)^2 + 4 = 4.$$

domain = $[-1, 3)$, codomain = $\mathbb{R}$, range of $f$ = $[4, 13)$.

*Remark.* If a function is given by a formula without domain specified, then assume domain = set of all $x$ for which $f(x)$ is well defined, this domain is also called the natural domain of $f$.

**Example 1.4.2.** Find the natural domain of the functions.

1. $f(x) = \dfrac{1}{x-3}$.
2. $g(t) = \dfrac{\sqrt{3-2t}}{t^2+4}$.

*Solution.*

1. $\dfrac{1}{x-3}$ is not defined when its denominator $x-3=0$, i.e. $x=3$. So the domain is $\mathbb{R}\backslash\{3\}$.
2. The domain of $\sqrt{3-2t}$ consists of all $x$ such that $3-2t \geq 0$, which implies that $t \leq \frac{3}{2}$. Hence the domain is $(-\infty, \frac{3}{2}]$.

■

**Example 1.4.3.** Let $f(x) = \dfrac{x^2-1}{x-1}$ and $g(x) = x+1$. Can we say $f$ and $g$ are the same function?

*Solution.* No! The domain of $f(x)$ is $\mathbb{R}\backslash\{1\}$, the domain of $g(x)$ is $\mathbb{R}$.
Only when $x \neq 1$, $f(x) = g(x)$. ■

### 1.4.1 Vertical Line Test for Graph

A way to visualize a function is its graph. If $f$ is a real-valued function of one variable, its graph consists of the points in the Cartesian plane $\mathbb{R}^2$ whose coordinates are the input-output pairs for $f$. In set notation, the graph is

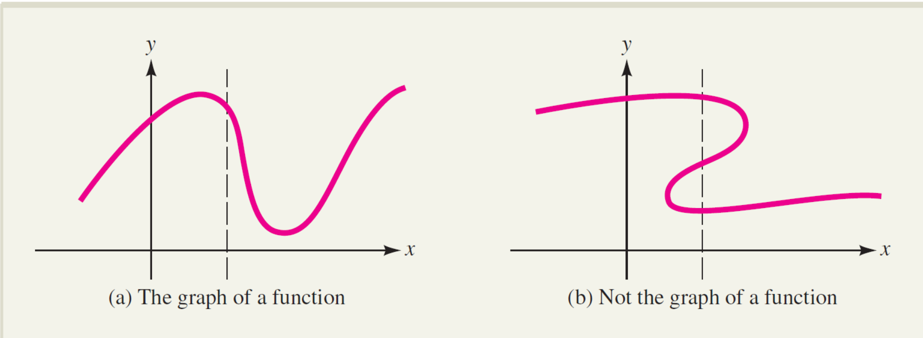
$$\{(x, y) \in \mathbb{R}^2 : x \in \mathbb{R}, y = f(x)\}.$$

**Review: Graphing a real-valued function of one variable:** [HBSP] 1.2.

**Example 1.4.4.** linear functions; piecewise linear functions; quadractic functions, exponential and log functions, trig functions.

It is important to realize that not every curve is the graph of a function. For instance, suppose the circle $x^2 + y^2 = 5$ were the graph of some function $y = f(x)$. Then, since the points $(1, 2)$ and $(1, -2)$ both lie on the circle, we would have $f(1) = 2$ and $f(1) = -2$, contrary to the requirement that a function assigns one and only one value to each number in its domain. Geometrically, this happens because the vertical line $x = 1$ intersects the graph of the circle more than once. The vertical line test is a geometric rule for determining whether a curve is the graph of a function.

**The Vertical Line Test** A curve is the graph of a function if and only if no vertical line intersects the curve more than once:



(a) The graph of a function (b) Not the graph of a function

### 1.4.2 Some Special Functions

**Definition 1.4.2.** A piecewise function is defined by more than one formula, with each individual formula defined on a subset of the domain.

**Example 1.4.5.** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

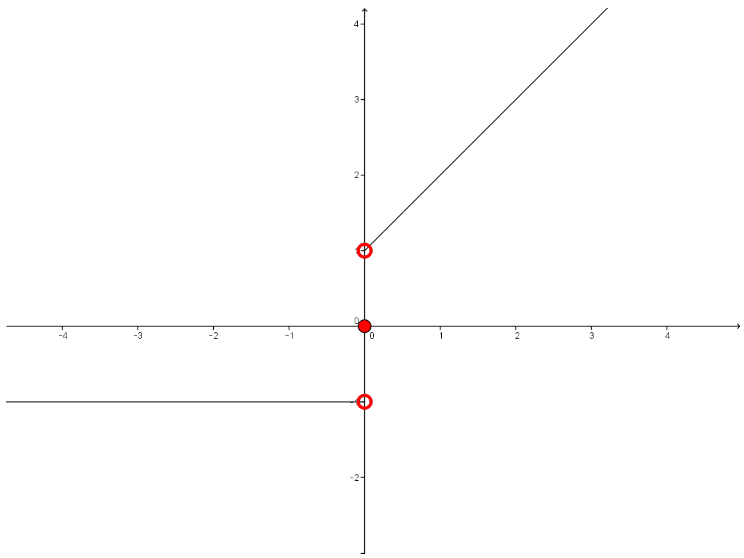
$$f(x) = \begin{cases} 1, & \text{if } x < 0 \\ 2x, & \text{if } x \geq 0. \end{cases}$$

Then $f(-1) = 1$, $f(0) = 0$ and $f(1) = 2$.

**Example 1.4.6.** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

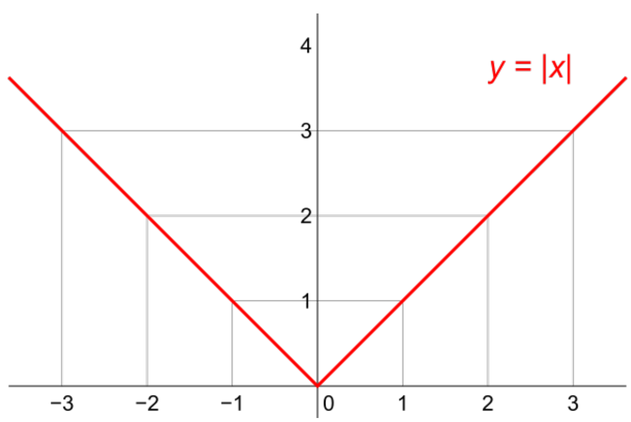
$$f(x) = \begin{cases} x+1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}$$

Then $f$ is a piecewise function.




**Example 1.4.7.** The absolute value function

$$|x| := \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$



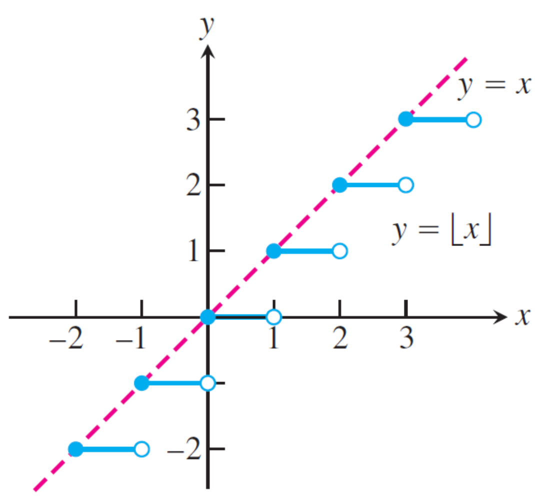
**Example 1.4.8.** Write $f(x) = 2x + |2 - x|$ as a piecewise function.

*Solution.* Note that $|2 - x| = 2 - x$ when $2 - x \geq 0$, that is $x \leq 2$; and $|2 - x| = x - 2$ when $2 - x < 0$, that is, $x > 2$. Hence $f(x) = 2x + 2 - x = x + 2$ if $x \leq 2$, and $f(x) = 2x + x - 2 = 3x - 2$ if $x > 2$, or we can write

$$f(x) = \begin{cases} x + 2 & \text{if } x \leq 2 \\ 3x - 2 & \text{if } x > 2 \end{cases}.$$

■

**Example 1.4.9.** Define the *floor function* as $\lfloor x \rfloor =$ the largest integer $\leq x$. Then $f(x) = \lfloor x \rfloor$ is a piecewise function.




*Exercise* 1.4.1. Define the *ceiling function* as $\lceil x \rceil =$ the smallest integer $\geq x$. Sketch the graph of $\lceil x \rceil$.

*Exercise* 1.4.2. Sketch the graph of

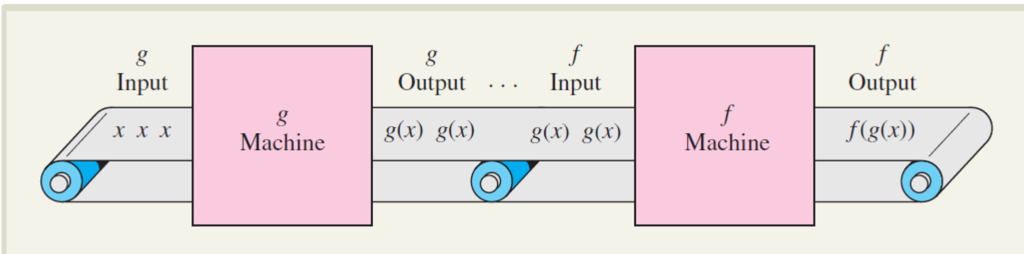
$$f(x) = \begin{cases} x - 2, & \text{if } x > 1, \\ -1, & \text{if } 0 \leq x \leq 1, \\ x^2, & \text{if } x < 0. \end{cases}$$

## 1.5 Composition of functions

**Definition 1.5.1.** Given functions $f(u)$ and $g(x)$, the composition of $f$ and $g$, denoted by $(f \circ g)(x)$, is a function of $x$ formed by substituting $u = g(x)$ for $u$ in the formula of $f(u)$, i.e.

$$(f \circ g)(x) = f(g(x)).$$

In the following figure, the definition of composite function is illustrated as an assembly line in which raw input $x$ is first converted into a transitional product $g(x)$ that acts as input in $f$ machine uses to produce $f(g(x))$.




**Example 1.5.1.** $f(x) = x^2 + 3x + 1$ and $g(x) = x + 1$.
Then

$$\begin{aligned}(f \circ g)(x) = f(g(x)) &= (g(x))^2 + 3(g(x)) + 1 = (x+1)^2 + 3(x+1) + 1 \\ &= (x^2 + 2x + 1) + (3x + 3) + 1 = x^2 + 5x + 5\end{aligned}$$

Similarly,

$$(g \circ f)(x) = g(f(x)) = f(x) + 1 = x^2 + 3x + 2.$$

*Remark.* In general, $(f \circ g)(x) \neq (g \circ f)(x)$.

**Example 1.5.2.** Suppose $f(x) = x^3 - 1$ and $h(x) = x - 1$, find $g(u)$ such that $f(x) = g(h(x))$.

*Solution.*

$$f(x) = x^3 - 1 = (x - 1 + 1)^3 - 1 = (x-1)^3 + 3(x-1)^2 + 3(x-1) = g(u),$$

where we define

$$g(u) = u^3 + 3u^2 + 3u.$$

■

*Alternative solution (change of variables):* Set $u = h(x)$ to be the new variable. Then $u = x - 1$ and $x$ may be expressed in terms of the new variable $u$ as $x = u + 1$. Plugging this into the formula for $f$, we have:

$$g(u) = f(x) = (u + 1)^3 - 1.$$

**Example 1.5.3.** Suppose $f(x) = (x - 5)^2 + \frac{3}{(x-5)^3}$, find $g(u)$ and $h(x)$ such that $f(x) = g(h(x))$.

*Solution.* The form of the given function is

$$f(x) = \square^2 + \frac{3}{\square^3},$$

where each box contains the expression $x - 5$. Thus $f(x) = g(h(x))$, where

$$g(u) = u^2 + \frac{3}{u^3} \text{ and } h(x) = x - 5.$$

■

*Remark.* There are many possible answers to the preceding problem, as $h(x)$ can be chosen quite arbitrarily. E.g. one may choose the new variable $u = h(x) = x - 1$, then $x = u + 1$ and

$$g(u) = f(x) = (u - 4)^2 + \frac{3}{(u-4)^3}.$$

**Definition 1.5.2.** A difference quotient for a function $f(x)$ is a composition function of the form

$$\frac{f(x+h) - f(x)}{h}$$

where $h$ is a constant.

Difference quotients are used to compute the slope of a tangent line to the graph and define the derivative, a concept of central importance in calculus.

**Example 1.5.4.** Find the difference quotient of $f(x) = x^2 - 3x$.

*Solution.*

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{[(x+h)^2 - 3(x+h)] - [x^2 - 3x]}{h} \\ &= \frac{[x^2 + 2xh + h^2 - 3x - 3h] - [x^2 - 3x]}{h} \\ &= \frac{2xh + h^2 - 3h}{h} = 2x + h - 3.\end{aligned}$$

■

**Geometric interpretation:** As slopes of secant lines to the graph of $f$.

$h \to 0 \rightsquigarrow$ tangent lines. Slopes of tangent lines to the graph of $f \rightsquigarrow$ derivatives of $f$.

## 1.6 Modeling in Business and Economics

**Example 1.6.1.** A manufacturer can produce dinning room tables at a cost of \$200 each. The table has been selling for \$300 each, and at that price consumers have been buying 400 tables per month. The manufacturer is planning to raise the price of the table and estimates that for each \$1 increase in the price, 2 fewer tables will be sold each month. What price corresponds to the maximum profit, and what is the maximum profit?

*Solution.* Let $x$ be the price.

$$\begin{aligned}
\text{Profit for one table} &= x - 200 \\
\text{Number of tables sold} &= 400 - 2(x-300) = 1000 - 2x \\
\text{Total profit:} \quad f(x) &= (x-200)(1000-2x) \\
&= -2x^2 + 1400x - 200000 \\
&= -2(x-350)^2 + 45000
\end{aligned}$$

$f(x)$ is maximized when the manufacturer charges \$350 for each table. ■

**Question**: How to find max/min for general functions? Calculus helps!